

ON A COMBINATORIAL PROBLEM OF ASMUS SCHMIDT

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ABSTRACT. For any integer $r \geq 2$, define a sequence of numbers $\{c_k^{(r)}\}_{k=0,1,\dots}$, independent of the parameter n , by

$$\sum_{k=0}^n \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \dots$$

We prove that all the numbers $c_k^{(r)}$ are integers.

1. STATING THE PROBLEM

The following curious problem was stated by A. L. Schmidt in [Sc1] in 1992.

Problem 1. *For any integer $r \geq 2$, define a sequence of numbers $\{c_k^{(r)}\}_{k=0,1,\dots}$, independent of the parameter n , by*

$$\sum_{k=0}^n \binom{n}{k}^r \binom{n+k}{k}^r = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} c_k^{(r)}, \quad n = 0, 1, 2, \dots \quad (1)$$

Is it then true that all the numbers $c_k^{(r)}$ are integers?

An affirmative answer for $r = 2$ was given in 1992 (but published a little bit later), independently, by Schmidt himself [Sc2] and by V. Strehl [St]. They both proved the following explicit expression:

$$c_n^{(2)} = \sum_{j=0}^n \binom{n}{j}^3 = \sum_j \binom{n}{j}^2 \binom{2j}{n}, \quad n = 0, 1, 2, \dots, \quad (2)$$

which was observed experimentally by W. Deuber, W. Thumser and B. Voigt. In fact, Strehl used in [St] the corresponding identity as a model for demonstrating various

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proof techniques of binomial identities. He also proved an explicit expression for the sequence $c_n^{(3)}$, thus answering affirmatively to Problem 1 in the case $r = 3$. But for this case Strehl had only one proof based on Zeilberger's algorithm of creative telescoping. Problem 1 was restated in [GKP] (the last Research Problem on p. 256) with indication (on p. 549) that H. S. Wolf had shown the desired integrality of $c_n^{(r)}$ for any r but only for any $n \leq 9$.

We recall that the first non-trivial case $r = 2$ is deeply related to the famous Apéry numbers $\sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$, the denominators of rational approximations to $\zeta(3)$. These numbers satisfy a 2nd-order polynomial recursion discovered by R. Apéry in 1978, while an analogous recursion (also 2nd-order and polynomial) for the numbers (2) was indicated by J. Franel already in 1894.

The aim of this paper is to give an answer in the affirmative to Problem 1 (Theorem 1) by deriving explicit expressions for the numbers $c_n^{(r)}$, and also to prove a stronger result (Theorem 2) conjectured in [St, Section 4.2].

Theorem 1. *The answer to Problem 1 is affirmative. In particular, we have the explicit expressions*

$$c_n^{(4)} = \sum_j \binom{2j}{j}^3 \binom{n}{j} \sum_k \binom{k+j}{k-j} \binom{j}{n-k} \binom{k}{j} \binom{2j}{k-j}, \quad (3)$$

$$c_n^{(5)} = \sum_j \binom{2j}{j}^4 \binom{n}{j}^2 \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j}, \quad (4)$$

and in general for $s = 1, 2, \dots$

$$\begin{aligned} c_n^{(2s)} &= \sum_j \binom{2j}{j}^{2s-1} \binom{n}{j} \sum_{k_1} \binom{j}{n-k_1} \binom{k_1}{j} \binom{k_1+j}{k_1-j} \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j}^2 \dots \\ &\quad \times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j}^2 \binom{2j}{k_{s-1}-j}, \\ c_n^{(2s+1)} &= \sum_j \binom{2j}{j}^{2s} \binom{n}{j}^2 \sum_{k_1} \binom{2j}{n-k_1} \binom{k_1+j}{k_1-j}^2 \sum_{k_2} \binom{2j}{k_1-k_2} \binom{k_2+j}{k_2-j}^2 \dots \\ &\quad \times \sum_{k_{s-1}} \binom{2j}{k_{s-2}-k_{s-1}} \binom{k_{s-1}+j}{k_{s-1}-j}^2 \binom{2j}{k_{s-1}-j}, \end{aligned}$$

where $n = 0, 1, 2, \dots$.

2. VERY-WELL-POISED PRELIMINARIES

The right-hand side of (1) defines the so-called *Legendre transform* of the sequence $\{c_k^{(r)}\}_{k=0,1,\dots}$. In general, if

$$a_n = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} c_k = \sum_{k=0}^n \binom{2k}{k} \binom{n+k}{n-k} c_k,$$

then by the well-known relation for inverse Legendre pairs one has

$$\binom{2n}{n} c_n = \sum_k (-1)^{n-k} d_{n,k} a_k,$$

where

$$d_{n,k} = \binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1} \binom{2n}{n-k}.$$

Therefore, putting

$$t_{n,j}^{(r)} = \sum_{k=j}^n (-1)^{n-k} d_{n,k} \binom{k+j}{k-j}^r, \quad (5)$$

we obtain

$$\binom{2n}{n} c_n^{(r)} = \sum_{j=0}^n \binom{2j}{j}^r t_{n,j}^{(r)}. \quad (6)$$

The case $r = 1$ of Problem 1 is trivial (that is why it is not included in the statement of the problem), while the cases $r = 2$ and $r = 3$ are treated in [Sc2], [St] using the fact that $t_{n,j}^{(2)}$ and $t_{n,j}^{(3)}$ have a *closed form*. Namely, it is easy to show by Zeilberger's algorithm of creative telescoping [PWZ] that the latter sequences, indexed by either n or j , satisfy simple 1st-order polynomial recursions. Unfortunately, this argument does not exist for $r \geq 4$.

V. Strehl observed in [St, Section 4.2] that the desired integrality would be a consequence of the divisibility of the product $\binom{2j}{j}^r \cdot t_{n,j}^{(r)}$ by $\binom{2n}{n}$ for all j , $0 \leq j \leq n$. He conjectured a much stronger property, which we are now able to prove.

Theorem 2. *The numbers $\binom{2n}{n}^{-1} \binom{2j}{j}^r t_{n,j}^{(r)}$ are integers.*

Our general strategy of proving Theorem 2 (and hence Theorem 1) is as follows: rewrite (5) in a hypergeometric form and apply suitable summation and transformation formulae (Propositions 1 and 2 below).

Changing l to $n - k$ in (5) we obtain

$$t_{n,j}^{(r)} = \sum_{l \geq 0} (-1)^l \frac{2n - 2l + 1}{2n - l + 1} \binom{2n}{l} \binom{n-l+j}{n-l-j}^r,$$

where the series on the right terminates. It is convenient to write all such terminating sums simply as \sum_l , which is, in fact, a standard convention (see, e.g., [PWZ]). The ratio of the two consecutive terms in the latter sum is equal to

$$\frac{-(2n+1)+l}{1+l} \cdot \frac{-\frac{1}{2}(2n-1)+l}{-\frac{1}{2}(2n+1)+l} \cdot \left(\frac{-(n-j)+l}{-(n+j)+l} \right)^r,$$

hence

$$t_{n,j}^{(r)} = \binom{n+j}{n-j}^r \cdot {}_{r+2}F_{r+1} \left(\begin{matrix} -(2n+1), -\frac{1}{2}(2n-1), -(n-j), \dots, -(n-j) \\ -\frac{1}{2}(2n+1), -(n+j), \dots, -(n+j) \end{matrix} \middle| 1 \right)$$

is a very-well-poised hypergeometric series. (We will omit the argument $z = 1$ in further discussions.)

The following two classical results—Dougall's summation of a ${}_5F_4(1)$ -series (proved in 1907) and Whipple's transformation of a ${}_7F_6(1)$ -series (proved in 1926)—will be required to treat the cases $r = 3, 4, 5$ of Theorems 1 and 2.

Proposition 1 [Ba, Section 4.3]. *We have*

$${}_5F_4 \left(\begin{matrix} a, 1 + \frac{1}{2}a, & c, & d, & -m \\ & \frac{1}{2}a, & 1 + a - c, & 1 + a - d, 1 + a + m \end{matrix} \right) = \frac{(1+a)_m (1+a-c-d)_m}{(1+a-c)_m (1+a-d)_m} \quad (7)$$

and

$$\begin{aligned} & {}_7F_6 \left(\begin{matrix} a, 1 + \frac{1}{2}a, & b, & c, & d, & e, & -m \\ & \frac{1}{2}a, & 1 + a - b, & 1 + a - c, & 1 + a - d, & 1 + a - e, 1 + a + m \end{matrix} \right) \\ &= \frac{(1+a)_m (1+a-d-e)_m}{(1+a-d)_m (1+a-e)_m} \cdot {}_4F_3 \left(\begin{matrix} 1 + a - b - c, & d, & e, & -m \\ & 1 + a - b, & 1 + a - c, & d + e - a - m \end{matrix} \right), \end{aligned} \quad (8)$$

where m is a non-negative integer.

Application of (7) gives (without creative telescoping)

$$t_{n,j}^{(3)} = \binom{n+j}{n-j}^3 \cdot \frac{(-2n)_{n-j} (-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2} = \frac{(2n)!}{(3j-n)! (n-j)!^3},$$

which is exactly the expression obtained in [St, Section 4.2]. Therefore, from (6) we have the explicit expression

$$c_n^{(3)} = \binom{2n}{n}^{-1} \sum_j \binom{2j}{j}^3 \frac{(2n)!}{(3j-n)! (n-j)!^3} = \sum_j \binom{2j}{j}^2 \binom{2j}{n-j} \binom{n}{j}^2.$$

For the case $r = 5$, we are able to apply the transformation (8):

$$\begin{aligned} t_{n,j}^{(5)} &= \binom{n+j}{n-j}^5 \cdot \frac{(-2n)_{n-j} (-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -2j, & -(n-j), & -(n-j), & -(n-j) \\ & -(n+j), & -(n+j), & 3j-n+1 \end{matrix} \right) \\ &= \binom{n+j}{n-j}^2 \frac{(2n)!}{(3j-n)! (n-j)!^3} \sum_l \frac{(-2j)_l (-n-j)_l^3}{l! (-n+j)_l^2 (3j-n+1)_l} \\ &= \frac{(2n)!}{(2j)! (n-j)!^2} \sum_l \binom{n-l+j}{n-l-j}^2 \binom{2j}{l} \binom{2j}{n-l-j} \\ &= \frac{(2n)!}{(2j)! (n-j)!^2} \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j}, \end{aligned}$$

hence

$$\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(5)} = \binom{n}{j}^2 \sum_k \binom{k+j}{k-j}^2 \binom{2j}{n-k} \binom{2j}{k-j}$$

are integers and from (6) we derive formula (4).

To proceed in the case $r = 4$, we apply the version of formula (8) with $b = (1+a)/2$ (so that the series on the left reduces to a ${}_6F_5(1)$ -very-well-poised series):

$$\begin{aligned}
t_{n,j}^{(4)} &= \binom{n+j}{n-j}^4 \cdot \frac{(-2n)_{n-j}(-2n+2(n-j))_{n-j}}{(-2n+(n-j))_{n-j}^2} \\
&\quad \times {}_4F_3 \left(\begin{matrix} -j, -(n-j), -(n-j), -(n-j) \\ -n, -(n+j), 3j-n+1 \end{matrix} \right) \\
&= \binom{n+j}{n-j} \frac{(2n)!}{(3j-n)!(n-j)!^3} \sum_l \frac{(-j)_l (-(n-j))_l^3}{l! (-n)_l (-(n+j))_l (3j-n+1)_l} \\
&= \frac{(2n)! j!}{n! (n-j)! (2j)!} \sum_l \binom{n-l+j}{n-l-j} \binom{j}{l} \binom{n-l}{j} \binom{2j}{n-l-j} \\
&= \frac{(2n)! j!}{n! (n-j)! (2j)!} \sum_k \binom{k+j}{k-j} \binom{j}{n-k} \binom{k}{j} \binom{2j}{k-j},
\end{aligned}$$

from which, again, $\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(4)} \in \mathbb{Z}$ and we arrive at formula (3).

3. ANDREWS'S MULTIPLE TRANSFORMATION

It seems that ‘classical’ hypergeometric identities can cover only the cases¹ $r = 2, 3, 4, 5$ of Theorems 1 and 2. In order to prove the theorems in full generality, we will require a multiple generalization of Whipple’s transformation (8). The required generalization is given by G.E. Andrews in [An, Theorem 4]. After making the passage $q \rightarrow 1$ in Andrews’s theorem, we arrive at the following result.

Proposition 2. *For $s \geq 1$ and m a non-negative integer,*

$$\begin{aligned}
& {}_{2s+3}F_{2s+2} \left(\begin{matrix} a, 1 + \frac{1}{2}a, & b_1, & c_1, & b_2, & c_2, & \dots \\ \frac{1}{2}a, & 1 + a - b_1, 1 + a - c_1, 1 + a - b_2, 1 + a - c_2, \dots \\ & \dots, & b_s, & c_s, & -m \\ & \dots, & 1 + a - b_s, 1 + a - c_s, 1 + a + m \end{matrix} \right) \\
&= \frac{(1+a)_m (1+a-b_s-c_s)_m}{(1+a-b_s)_m (1+a-c_s)_m} \sum_{l_1 \geq 0} \frac{(1+a-b_1-c_1)_{l_1} (b_2)_{l_1} (c_2)_{l_1}}{l_1! (1+a-b_1)_{l_1} (1+a-c_1)_{l_1}} \\
&\quad \times \sum_{l_2 \geq 0} \frac{(1+a-b_2-c_2)_{l_2} (b_3)_{l_1+l_2} (c_3)_{l_1+l_2}}{l_2! (1+a-b_2)_{l_1+l_2} (1+a-c_2)_{l_1+l_2}} \dots \\
&\quad \times \sum_{l_{s-1} \geq 0} \frac{(1+a-b_{s-1}-c_{s-1})_{l_{s-1}} (b_s)_{l_1+\dots+l_{s-1}} (c_s)_{l_1+\dots+l_{s-1}}}{l_{s-1}! (1+a-b_{s-1})_{l_1+\dots+l_{s-1}} (1+a-c_{s-1})_{l_1+\dots+l_{s-1}}} \\
&\quad \times \frac{(-m)_{l_1+\dots+l_{s-1}}}{(b_s+c_s-a-m)_{l_1+\dots+l_{s-1}}}.
\end{aligned}$$

Proof of Theorem 2. As in Section 2, we will distinguish the cases corresponding to the parity of r .

¹This is not really true since Andrews’s ‘non-classical’ identity below is a consequence of very classical Whipple’s transformation and the Pfaff–Saalschütz formula.

If $r = 2s + 1$, then setting $a = -(2n + 1)$ and $b_1 = c_1 = \dots = b_s = a_s = -m = -(n - j)$ in Proposition 2 we obtain

$$\begin{aligned}
t_{n,j}^{(2s+1)} &= \binom{n+j}{n-j}^{2s-2} \frac{(2n)!}{(3j-n)!(n-j)!^3} \sum_{l_1} \binom{2j}{l_1} \left(\frac{(-(n-j))_{l_1}}{(-(n+j))_{l_1}} \right)^2 \\
&\quad \times \sum_{l_2} \binom{2j}{l_2} \left(\frac{(-(n-j))_{l_1+l_2}}{(-(n+j))_{l_1+l_2}} \right)^2 \dots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \left(\frac{(-(n-j))_{l_1+\dots+l_{s-1}}}{(-(n+j))_{l_1+\dots+l_{s-1}}} \right)^2 \\
&\quad \times \frac{(-1)^{l_1+\dots+l_{s-1}}(-(n-j))_{l_1+\dots+l_{s-1}}}{(3j-n+1)_{l_1+\dots+l_{s-1}}} \\
&= \frac{(2n)!}{(2j)!(n-j)!^2} \sum_{l_1} \binom{2j}{l_1} \binom{n-l_1+j}{n-l_1-j}^2 \sum_{l_2} \binom{2j}{l_2} \binom{n-l_1-l_2+j}{n-l_1-l_2-j}^2 \dots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \binom{n-l_1-\dots-l_{s-1}+j}{n-l_1-\dots-l_{s-1}-j}^2 \cdot \binom{2j}{n-l_1-\dots-l_{s-1}-j}.
\end{aligned}$$

If $r = 2s$, we apply Proposition 2 with the choice $a = -(2n+1)$, $b_1 = (a+1)/2 = -n$ and $c_1 = b_2 = \dots = b_s = a_s = -m = -(n - j)$:

$$\begin{aligned}
t_{n,j}^{(2s)} &= \binom{n+j}{n-j}^{2s-3} \frac{(2n)!}{(3j-n)!(n-j)!^3} \sum_{l_1} \binom{j}{l_1} \frac{(-(n-j))_{l_1}}{(-n)_{l_1}} \frac{(-(n-j))_{l_1}}{(-(n+j))_{l_1}} \\
&\quad \times \sum_{l_2} \binom{2j}{l_2} \left(\frac{(-(n-j))_{l_1+l_2}}{(-(n+j))_{l_1+l_2}} \right)^2 \dots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \left(\frac{(-(n-j))_{l_1+\dots+l_{s-1}}}{(-(n+j))_{l_1+\dots+l_{s-1}}} \right)^2 \\
&\quad \times \frac{(-1)^{l_1+\dots+l_{s-1}}(-(n-j))_{l_1+\dots+l_{s-1}}}{(3j-n+1)_{l_1+\dots+l_{s-1}}} \\
&= \frac{(2n)! j!}{n! (n-j)! (2j)!} \sum_{l_1} \binom{j}{l_1} \binom{n-l_1}{j} \binom{n-l_1+j}{n-l_1-j} \\
&\quad \times \sum_{l_2} \binom{2j}{l_2} \binom{n-l_1-l_2+j}{n-l_1-l_2-j}^2 \dots \\
&\quad \times \sum_{l_{s-1}} \binom{2j}{l_{s-1}} \binom{n-l_1-\dots-l_{s-1}+j}{n-l_1-\dots-l_{s-1}-j}^2 \cdot \binom{2j}{n-l_1-\dots-l_{s-1}-j}.
\end{aligned}$$

In both cases, the desired integrality

$$\binom{2n}{n}^{-1} \binom{2j}{j} t_{n,j}^{(r)} \in \mathbb{Z}, \quad j = 0, 1, \dots, n,$$

clearly holds, and Theorem 2 follows.

Theorem 1 is an immediate consequence of Theorem 2.

We would like to conclude the paper by the following q -question.

Problem 2. *Find and solve an appropriate q -analogue of Problem 1.*

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